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SPECIAL ARTICLE

An Introduction to the Article "Reminiscences about Difference Schemes" by S. K. Godunov

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Some footnotes are given to the keynote address given by the Russian mathematician S. K. Godunov at a symposium in his honor, held in May 1997 at the University of Michigan.

Key Words: hyperbolic equations; finite-volume method.

1. INTRODUCTION

On May 1–2, 1997, an international symposium was held in honor of the Russian mathematician Sergei Konstantinovich Godunov. The title of the symposium was "Godunov's Method for Gas Dynamics: Current Applications and Future Developments," and its venue was the Department of Aerospace Engineering at the University of Michigan, Ann Arbor, MI. The meeting preceded the awarding of an honorary doctorate to Godunov, which took place during the University's Spring Commencement, on May 3.

Although the *Journal of Computational Physics* rarely pays attention to conferences, the Godunov Symposium was an exception. It was announced in this journal in a Letter to the Editor [1] by the current author, followed by an Erratum [2] that contained the true image of Godunov. Moreover, the *Journal* shared sponsorship of the symposium with the University of Michigan, AFOSR, and NSF, by hosting one luncheon.

This involvement reflects the special relationship between the *Journal of Computational Physics and Godunov-type* methods. Quoting from the letter [1]:

> ... no single journal has done so much for the advancement of Godunov-type methods as JCP. If we define Godunov-type methods as non-oscillatory finite-volume schemes that incorporate the solution (exact or approximate) to Riemann's initial-value problem, or a generalization of it, we find that almost all key articles on the development of such methods appeared in this journal. And among the

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authors of these papers several have served or are serving on its editorial board. [NB: B. van Leer, A. J. Chorin, P. L. Roe, S. Osher, and R. J. LeVeque.]

In continuation of this special relationship, this issue of JCP offers an English translation of the keynote lecture presented by Godunov himself at the symposium [3]. The latter paper contains a wealth of historic and anecdotal information regarding the early years of his career, when he worked in CFD; yet, it is useful to supplement it by some notes on the significance of that work. This is done in the next section; after that follows a perspective on Godunov-type methods as it emerged from the symposium's invited lectures and panel discussions.

2. GODUNOV AND HIS WORK IN CFD

S. K. Godunov, born July 17, 1929, in Russia, studied mathematics at Moscow State University under such illustrious tutors as I. M. Gel'fand, I. G.Petrovskii, and M. V. Keldish, receiving his doctoral degree in 1954. The early years of his career he spent in Moscow at what is now called the Keldish Institute of Mathematics. In 1969 he moved to the Siberian Branch of the Soviet Academy of Science in Akademgorodok near Novosibirsk, where he still lives and works. For many years he held positions in the Computing Center, while also teaching at Novosibirsk State University. He currently is Head of the Department of Partial Differential Equations in the Sobolev Institute of Mathematics. Godunov became a corresponding member of the Soviet Academy of Science in 1976 and has been a full member of the Russian Academy since 1994.

Godunov's work of the 1950s and 1960s (the Moscow period) was in the field of hyperbolic partial differential equations (PDEs) and their numerical approximation and has had an profound effect on computational fluid dynamics (CFD). Many of today's state-of-the-art codes for simulating compressible flow, used in fields as diverse as semiconductor modeling, civil and military aeronautics, general circulation modeling, planetary space physics, and relativistic astrophysics, have their roots in a single paper published by Godunov [4] in 1959, based on his Ph.D. thesis.

In this paper Godunov develops a unique physical style of thinking about numerically simulating gas flow, a style that has remained inspiring and effective to this day. Godunov's line of reasoning leads him to adopt the solution to Riemann's initial-value problem as a building block for his finite-volume method for compressible flow; the current CFD-lingo "Riemann solver" indicates this has become standard practice.

The paper further includes "Godunov's Theorem," taught today to every student of CFD; it states that any monotonicity-preserving difference scheme for the linear advection equation can be no better than first-order accurate. This implies an ever-present accuracy barrier for flow computations; how to circumvent it through the use of nonlinear techniques was a major breakthrough in the early 1970s [5, 6]. This opened the door to the development and use of Godunov-type methods in an ever-expanding range of disciplines of science and engineering, for computing continuum processes dominated by wave propagation. The breadth and depth of the collective research effort that brought about this numerical revolution are well captured in the 1997 anthology "Upwind and high-resolution schemes," edited by Hussaini, van Leer, and van Rosendale [7]. It contains 21 annotated reprints of key articles on the subject, preceded by a historical account by the current author, and a technical introduction by P. L. Roe.

Another strong contribution to numerical mathematics is Godunov's work with Rjabenkii [8] on the stability of finite-difference schemes in the presence of boundaries; this became the basis of further analysis and the development of practical stability criteria by Osher [9] and, ultimately. Gustafsson, Kreiss, and Sundström [10].

Godunov's best-known non-numerical contribution is his observation, made in a short 1961 paper [11], that the symmetrizability of many hyperbolic systems describing natural phenomena implies the existence of an entropy function. Such a function can be used to enforce irreversibility of discontinuous solutions, whether exact or approximate; see, e.g., [12]. The property of symmetrizability of a hyperbolic system may get lost in manipulating the system for numerical purposes, for instance, after adding terms to establish the conservation form [13, 14]. Owing to Godunov, today's numerical analysts are well aware of the benefits of the symmetrizable form.

With the move to Novosibirsk (1969), Godunov left CFD. For many years his research focused on the mechanics and thermodynamics of continuous media, in particular, on the theory of plasticity. His current research interest is in computational linear algebra. He published a book [15] in English in 1993, titled "Guaranteed Accuracy in Numerical Linear Algebra"; his latest book [16], "Modern Aspects of Linear Algebra," has also appeared in English translation.

3. A PERSPECTIVE ON GODUNOV-TYPE METHODS

The first symposium on Godunov-type methods covered a range of disciplines spanning all scales, from semiconductor modeling to relativistic astrophysics. Symposium participants came away with a strong sense of the generality and uniform applicability of Godunov-type methods. In all fairness, though, it must be said that the limitations of the approach were also indicated and discussed.

The key question was posed by Charles Hirsch (Free University Brussels) during the last panel discussion: How far can and should one still go in analyzing the detailed wave structure of physical systems, for use in a Godunov-type method? To put this question in a perspective, some background is needed.

Godunov-type methods are successfully used world-wide to model continuum processes dominated by wave motion, in particular, those described by hyperbolic systems of conservation laws. Central to the success of these methods is the use of the solution to Riemann's initial-value problem, which describes the interaction through waves of two abutting parcels of the medium. While this premise is undisputed, much effort has gone into searching how far one can go in dispensing with the accuracy of the Riemann solution, and thereby with its computational complexity, without fatally compromising the accuracy of the overall method. Godunov, Zabrodyn, and Prokopov [17] already introduced a linearized Riemann solver for smooth flow in their two-dimensional Euler method.

Van Leer [18] approximated the numerical flux of Godunov's method by a finite-difference expression similar to the one used in the Lax–Wendroff scheme. This flux was greatly improved by Roe's local linearization [19] of the equation system, which captures the wave structure not just in the limit of weak waves but also in the limit of one dominant wave. Osher's approximate Riemann solver [20] is based on simple waves only. All these approximations still recognize every single wave present in the fundamental interaction.

This approach is abandoned in the approximate Riemann solver of Harten, Lax, and van Leer [12], which distinguishes only three distinct wave speeds, regardless of the actual number of waves. Such a Riemann solver has been developed by symposium speaker Linde (University of Michigan) [21] to approximate the 8-wave system of magneto-hydrodynamics. Larger and more intricate hyperbolic systems arise when taking higher moments of the collisional Boltzmann equation, as used by symposium speaker Groth (University of Michigan) [22] to describe flow at intermediate Knudsen numbers. Their complex, hard-to-compute eigenvector structure and eigenvalues may be at or beyond the limit of what a practical computational scheme should include.

This brings us back to the original question. There are obvious rewards in first investigating the details of the wave system admitted by a hyperbolic system of conservation laws: one learns a great deal about the physics embedded in the equations and even obtains some analytical benchmark solutions, before attempting numerical integration. One might actually say this is the way it *should* be. However, the equation system may be so complex that its analysis seems unfeasible, even with aid of symbolic manipulation, or there may be a non-analytical component, such as a tabular equation of state. In such cases, obtaining some provisional numerical solutions might actually help one understand the physics and eventually carry out the wave analysis.

To get those benefits, one needs a numerical method that does not rely on a detailed solution of the Riemann problem. An example is the higher-order method of the Lax–Friedrichs type proposed by Nessyahu and Tadmor [23] and further developed by symposium panelist Arminjon (University of Montreal) [24], where grid staggering in time eliminates the Riemann problem. The formulation of such an integration scheme for a new, complicated, and unexplored set of equations requires almost zero start-up time. We may therefore expect this class of methods to gain importance as the physical systems dictated by science and engineering get more and more complicated.

However this may be, it is certain that physical reasoning will remain the starting point in the development of CFD methods. This we owe to S. K. Godunov.

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